Integral s parametrom

Naj bo $f(x,t)$: [a,b]×[c,d] → ℝ funkcija:

Zveznost

$f(x,t)$ zvezna na $D_f \implies$ F$(x) = \int_c^d f(x,t)dt$ zvezna na [a,b].

Odvedljivost

$f(x,t)$ zvezna in zvezno parcialno odvedljiva po x na $D_f \implies$ F$(x) = \int_c^d f(x,t)dt$ odvedljiva na [a,b] in velja:

$$F'(x) = \int_c^d f_x(x,t)dt.$$

Integrabilnost

$f(x,t)$ zvezna na $D_f \implies$ F$(x) = \int_c^d f(x,t)dt$ integrabilna na [a,b] in velja:

$$\int_a^b F(x)dx = \int_a^b (\int_c^d f(x,t)dt)dx = \int_c^d (\int_a^b f(x,t)dx)dt.$$

Integral z variabilnimi mejami

Zveznost

Naj bo $f(x,t)$: [a,b]×[c,d] → ℝ zvezna in u,v: [a,b] → [c,d] zvezni $\implies$ $F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$ zvezna na [a,b].

Odvedljivost

Naj bo $f(x,t)$: [a,b]×[c,d] → ℝ zvezna in zvezno parcialno odvedljiva po x na D_f in naj bosta u,v: [a,b] → [c,d] odvedljivi $\implies$
$F(x) = \int_{u(x)}^{v(x)} f(x,t)dt$ odvedljiva in velja:

$$F'(x) = \int_{u(x)}^{v(x)} f_x(x,t)dt + v'(x)f(x,v(x)) - u'(x)f(x,u(x)).$$

Izlimitirani integral s parametrom

Integral s parametrom $F(x) = \int_a^\infty f(x,t)dt$ je **enakomerno konvergenten** za x ∈ [c,d], če za $\forall\varepsilon > 0\ \exists b > a$, da velja:

$$\left|\int_b^\infty f(x,t)dt\right| < \varepsilon \quad \forall x \in [c,d].$$

Weierstrass M-test

Če $\exists g$: [a,∞) → ℝ, da velja $|f(x,t)| < g(t)$ za $\forall x \in$ [c,d] in je $\int_a^\infty g(t)dt < \infty \implies F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergenten na [c,d]. V pomoč sta formuli:

$$\int_0^s g(x)\ t^{-\alpha}dt < \infty\ s > 0 \iff \alpha < 1,\ g(x) \text{ omejena.}$$

$$\int_s^\infty g(x)\ t^{-\alpha}dt < \infty\ s > 0 \iff \alpha > 1,\ g(x) \text{ omejena.}$$

Naj bo $f(x,t)$: [c,d]×[a,∞) → ℝ funkcija:

Zveznost

$f(x,t)$ zvezna na D_f in $F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergentna na [c,d] $\implies$ F zvezna na [c,d].

Odvedljivost

$f(x,t)$ zvezna in zvezno parcialno odvedljiva po x na D_f, $F(x) = \int_a^\infty f(x,t)dt$ konvergentna na [c,d] ter $F(x) = \int_a^\infty f_x(x,t)dt$ enakomerno konvergentna na [c,d] $\implies F'(x) = \int_a^\infty f_x(x,t)dt$.

Integrabilnost

$f(x,t)$ zvezna na D_f in $F(x) = \int_a^\infty f(x,t)dt$ enakomerno konvergentna na [c,d], potem velja:

$$\int_c^d F(x)dx = \int_c^d (\int_a^\infty f(x,t)dt)dx = \int_a^\infty (\int_c^d f(x,t)dx)dt.$$

Gamma in Beta funkciji

Gamma funkcija:

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \quad x \in (0,\infty)$$

1. $\Gamma(x+1) = x\Gamma(x)\ \forall x > 0$
2. $\Gamma(n) = (n-1)!\ \forall n \in \mathbb{N}$
3. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Beta funkcija:

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt \quad p,q > 0$$

1. $\beta(p,q) = \frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}.\ \forall p,q > 0$
2. $\beta(p,q) = \int_0^\infty \frac{u^{p-1}}{(1+u)^{p+q}}du\ (t = \frac{u}{1+u})$
3. $\int_0^{\pi/2} sin^{2p-1}x \cdot cos^{2q-1}xdx = \frac{1}{2}\beta(p,q) \quad p,q > 0\ (t = \sin x^2)$
4. $\beta(1,q) = \frac{1}{q}$
5. $\beta(p+1,q) = \frac{p}{p+q}\beta(p,q)$
6. $\beta(p,q) = \beta(q,p) \quad \beta(\frac{1}{2},\frac{1}{2}) = \pi$
7. $\beta(p,1-p) = \frac{\pi}{sin(\pi p)}, \quad \frac{d}{d^{[k]}x}(\beta(p,1-p)) = \frac{u^x \ln^k(x)}{1+u}, \quad p \in (0,1)$

$\int_0^1 x^p(1-x)^q\ dx = \beta\,(p+1,q+1) = \frac{\Gamma(p+1)\Gamma(q+1)}{\Gamma(p+q+2)}$
$\int_0^\infty \frac{x^p}{(1+x)^q}\ dx = \beta\,(p+1,q-p-1) = \frac{\Gamma(p+1)\Gamma(q-p-1)}{\Gamma(q)}$
$\int_0^{\frac{\pi}{2}} \sin^p\varphi\cos^q\varphi\ d\varphi = \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})}$

Večterni integral

Fubini

(1) Če f integrabilna na [a,b]×[c,d]⊂ ℝ² in $x \mapsto f(x,y)$ integrabilna na [a,b] za $\forall y \in$[c,d] in $y \mapsto f(x,y)$ integrabilna na [c,d] za $\forall x \in$ [a,b], potem:

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy = \int_a^b (\int_c^d f(x,y)dy)dx = \int_c^d (\int_a^b f(x,y)dx)dy$$

Analogno za $n \geq 3$.
(2) Naj bo $\Omega = \{(x,y,z) \in \mathbb{R}^3 |\ (x,y) \in D,\ g(x,y) \leq z \leq b(x,y)\}$, potem:

$$\iiint_\Omega f(x,y,z)dxdydz = \iint_D (\int_{g(x,y)}^{b(x,y)} f(x,y,z)dz)dxdy$$

(3) Fubinijev izrek v posplošenem integralu $\iiint_\Omega f(x,y,z)dxdydz$ lahko uporabimo če:

- Ω omejeno območje in f omejena funkcija **ali**
- f pozitivna funkcija **ali**
- $\iiint_\Omega |f(x,y,z)|\ dxdydz < \infty$

Nove spremenljivke

Jacobijeva matrika

Naj bo $f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x_1,\ldots,x_n) = (f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n))$, Jacobijevo matriko definiramo kot:

$$J_{f(x_1,\ldots,x_n)} = \begin{bmatrix} f_{1_{x_1}} & f_{1_{x_2}} & \ldots & f_{1_{x_n}} \\ f_{2_{x_1}} & f_{2_{x_2}} & \ldots & f_{2_{x_n}} \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ f_{m_{x_1}} & f_{m_{x_2}} & \ldots & f_{m_{x_n}} \end{bmatrix}$$

Vpeljava

Naj bosta $\Omega \subseteq \mathbb{R}^2$ in $\varphi : \Omega \to \mathbb{R}^2$ preslikava z zvezno odvedljivimi komponentami. Naj bo $detJ_\varphi \neq 0$ na Ω in naj bo $f : \varphi(\Omega) \to \mathbb{R}^2$ zvezna. Potem:

$$\iint_{\varphi(\Omega)} f(x,y)dxdy = \iint_\Omega f(\varphi(t,s)) \cdot |detJ_\varphi(t,s)|\ dtds$$

Polarne koordinate

$x = r\cos\varphi \quad y = r\sin\varphi \quad |detJ| = r \quad r \geq 0 \quad \varphi \in [0,2\pi]$

Cilindrične koordinate

$x = r\cos\varphi \quad y = r\sin\varphi \quad z = z \quad |detJ| = r \quad r \geq 0 \quad \varphi \in [0,2\pi]$

Sferične koordinate

$x = R\cos\varphi\cos\vartheta \quad y = R\sin\varphi\cos\vartheta \quad z = R\sin\vartheta \quad R \geq 0 \quad \varphi \in [0,2\pi]$
kjer za ϑ velja:

- $\vartheta \in [-\frac{\pi}{2},\frac{\pi}{2}] \implies |detJ| = R^2\cos\vartheta$
- $\vartheta \in [0,\pi] \implies |detJ| = R^2\sin\vartheta$

Uporaba

$\rho(a)\ldots$ gostota v točki a, $D \subset \mathbb{R}^2$, $\Omega \subset \mathbb{R}^3$

Ploščina(S) ali volumen(V)

$S(D) = \iint_D dxdy \quad V(\Omega) = \iiint_\Omega dxdydz$

Masa(m)

$m(D) = \iint_D \rho(x,y)dxdy \quad m(\Omega) = \iiint_\Omega \rho(x,y,z)dxdydz$

Masno središče(ā)

$\bar{x} = \frac{1}{m(D)}\iint_D x\cdot\rho(x,y)dxdy \quad \bar{x} = \frac{1}{m(\Omega)}\iiint_\Omega x\cdot\rho(x,y,z)dxdydz$
$\bar{y} = \frac{1}{m(D)}\iint_D y\cdot\rho(x,y)dxdy \quad \bar{y} = \frac{1}{m(\Omega)}\iiint_\Omega y\cdot\rho(x,y,z)dxdydz$
$\bar{z} = \frac{1}{m(\Omega)}\iiint_\Omega z\cdot\rho(x,y,z)dxdydz$

Vztrajnostni moment (J)

okoli izhodišča: $J(D) = \iint_D (x^2+y^2)\cdot\rho(x,y)dxdy$

okoli z-osi: $J_z(\Omega) = \iiint_\Omega (x^2+y^2)\cdot\rho(x,y,z)dxdydz$

okoli y-osi: $J_y(\Omega) = \iiint_\Omega (x^2+z^2)\cdot\rho(x,y,z)dxdydz$

okoli x-osi: $J_x(\Omega) = \iiint_\Omega (y^2+z^2)\cdot\rho(x,y,z)dxdydz$

Konvergenca

$\iint_D f(x,y)dxdy$ je **absolutno konvergenten**, če konvergira tudi $\iint_D |f(x,y)|\, dxdy$.

V primeru da $\iint_D f(x,y)dxdy$ konvergira, $\iint_D |f(x,y)|\, dxdy$ pa ne, pravimo da je integral **pogojno konvergenten**.

(1) $\iint_D |f(x,y)|\, dxdy$ konvergenten $\implies$ $\iint_D f(x,y)dxdy$ konvergenten.

(2) $\exists \iint_{\mathbb{R}^2} |f(x,y)|\, dxdy$ ali $\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |f(x,y)|\, dy$ ali $\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |f(x,y)|\, dx$ potem $\exists \iint_{\mathbb{R}^2} f(x,y)dxdy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} f(x,y)dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} f(x,y)dx$.

Krivuljni in ploskovni integral

Krivulje

$\vec{r(t)} : [a,b] \to K, \quad \vec{r(t)} = (x(t),y(t),z(t))$

$\dot{\vec{r}}(t) = (\dot{x}^2,\dot{y}^2,\dot{z}^2), \quad \left|\dot{\vec{r}}(t)\right| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2)}$

$$\boldsymbol{I_1} = \int_K \rho ds = \int_a^b \rho(\vec{r}(t)) \left|\dot{\vec{r}}(t)\right| dt$$

$$\boldsymbol{I_2} = \int_K \vec{V} d\vec{s} = \int_K V_1 dx + V_2 dy + V_3 dz = \int_a^b V(\vec{r}(t))\dot{\vec{r}}(t)dt$$

dolžina krivulje: $l(K) = \int_a^b ds = \int_a^b \left|\dot{\vec{r}}(t)\right| dt$

lastnosti:

1. linearnost t.j. za $\alpha, \beta \in \mathbb{R}$ velja:
   $\int_\gamma (\alpha\rho_1 + \beta\rho_2)ds = \alpha \int_\gamma \rho_1 ds + \beta \int_\gamma \rho_2 ds$
   $\int_\gamma (\alpha\vec{V_1} + \beta\vec{V_2})d\vec{s} = \alpha \int_\gamma \vec{V_1} d\vec{s} + \beta \int_\gamma \vec{V_2} d\vec{s}$
2. če je $\gamma_1 \cap \gamma_2 = \emptyset$ za integrala obeh vrst velja
   $\int_\gamma = \int_{\gamma_1} + \int_{\gamma_2}$
3. drugi je odvisen od orientacije, prvi pa ne:
   $\boldsymbol{I_1} = \int_\gamma \rho ds = \int_{-\gamma} \rho ds \quad \boldsymbol{I_2} = \int_\gamma \vec{V} d\vec{s} = -\int_{-\gamma} \vec{V} d\vec{s}$

Ploskve

$\vec{r} : D \to P, \quad \vec{r(u,v)} = (x(u,v),y(u,v),z(u,v))$

$\vec{r}_u \times \vec{r}_v = (y_u z_v - y_v z_u, x_v z_u - x_u z_v, x_u y_v - x_v y_u)$
$|\vec{r}_u \times \vec{r}_v| = \sqrt{EG - F^2}, \quad E = \vec{r}_u \cdot \vec{r}_u, G = \vec{r}_v \cdot \vec{r}_v, F = \vec{r}_u \cdot \vec{r}_v$
$\vec{n} = \frac{(\vec{r}_u \times \vec{r}_v)}{|\vec{r}_u \times \vec{r}_v|}$

$$\boldsymbol{I_3} = \iint_P \rho dS = \int_P \rho(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dudv$$

$$\boldsymbol{I_4} = \iint_P \vec{V} d\vec{S} = \iint_P V_1 dydz + V_2 dzdx + V_3 dxdy = \iint_P [V(\vec{r}(u,v)), \vec{r}_u, \vec{r}_v] dudv = \iint_P V(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) dudv$$

površina ploskve: $pl(P) = \iint_P ds = \iint_P |\vec{r}_u \times \vec{r}_v|\, dudv$

vrtenina z-os:
$pl(P) = 2\pi \int_a^b \rho(t)\sqrt{\dot{\rho}^2(t) + \dot{z}^2(t)}dt \quad (x,z) = (\rho(t), z(t)) \subseteq [0,\infty) \times \mathbb{R}$

Kompleksna analiza

$z = e^{it} = cos(t) + isin(t)$
$z = x + iy \quad x = Re(z) = \frac{z+\overline{z}}{2}$
$\overline{z} = x - iy \quad y = Im(z) = \frac{z-\overline{z}}{2i}$
$cos(t) = \frac{e^{it}+e^{-it}}{2}$
$sin(t) = \frac{e^{it}-e^{-it}}{2i}$

Kompleksna funkcija kot realna preslikava:
$f(z) = u(z) + iv(z) = u(x + iy) + iv(x + iy)$

Ustrezna realna preslikava:
$f_{\mathbb{R}}(x,y) = (u(x,y), v(x,y)) : D \subset \mathbb{R}^2 \to \mathbb{R}^2 \cong u(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}) + iv(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}) = f(z)$

Integral funkcije $f : \mathbb{C} \to \mathbb{C}$ po γ definiramo kot:
$\int_\gamma f(z)dz = \int_a^b f(\rho(t)) * \dot{\rho(t)}dt,$
kjer je $\rho(t)$ parametrizacija krivulje γ in $*$ kompleksno množenje.

Zveznost

$f_{\mathbb{R}} = (u,v)$ **zvezna** v (a,b), če
$\forall\varepsilon > 0 \exists\delta > 0 \ni$: $\sqrt{(x - a)^2 + (y - b)^2} < \delta \Rightarrow \sqrt{(u(x,y) - u(a,b))^2 + (v(x,y) - v(a,b))^2} < \varepsilon$

$f = u + iv$ **zvezna** v $c = a + ib$, če
$\forall\varepsilon > 0 \exists\delta > 0 \ni$: $|z - c| < \delta \Rightarrow |f(z) - f(c)| < \varepsilon$
Oba pojma sta ekvivalentna.

Holomorfne funkcije

Funkcija $f : D \subset \mathbb{C} \to \mathbb{C}$ je **holomorfna** v $a \in D$ ali na D, če tam obstaja njen kompleksni odvod.
$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}; h \in \mathbb{C}$

Cauchy Reimannov sistem enačb (CRS): Naj bo $D \subset \mathbb{C}$ odprta in $f = u + iv : D \to \mathbb{C}$ kompleksna funkcija z zvezno odvedljivima u in v.
Potem je **f holomorfna** natanko tedaj, ko je:

$$u_x = v_y \quad in \quad u_y = -v_x$$

opomba: v praksi so holomorfne tiste kompleksne funkcije, pri katerih nastopa le z in ne tudi $\overline{z}$.

(Cauchyev izrek) Naj bo $D \subset \mathbb{C}$ območje brez lukenj in naj bo $\gamma \subset D$ sklenjena krivulja. Potem velja: $\int_\gamma f(z)dz = 0$.

Greenova formula: $\oint_{\partial\Omega} (P,Q)d\vec{s} = \iint_\Omega (Qy - Px)dxdy$

Če je $D \subset \mathbb{C}$ območje brez lukenj, $f : D \to \mathbb{C}$ holomorfna in $\gamma_1, \gamma_2 \subset D$ dve poti, ki se končata in začneta v isti točki, potem je $\int_{\gamma_1} fdz = \int_{\gamma_2} fdz$
t.j. integral holomorfne funkcije je neodisen od izbire poti.

Kompleksne krivulje

$\gamma : [a,b] \to K, \quad \gamma = x(t) + iy(t)$
$\int_K f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$

Izrek o residuih

za holomorfno funkcijo $f : D\backslash\{a_1, \ldots ,a_n\} \to \mathbb{C}$ velja

$$\oint_{\partial D} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f,a_k)$$

kjer je ∂D rob območja D

f ima pol 1. stopnje: $\quad Res(f,a_k) = \lim_{z\to a} (z - a)f(z)$
f ima pol 2. stopnje: $\quad Res(f,a_k) = \lim_{z\to a} ((z - a)^2 f(z))'$

Triki za računanje realnih integralov

1. z vpeljavo nove spremenljivke $z = e^{it}$ dobimo zvezo

$$\int_0^{2\pi} f(cos(t),sin(t))\; dt = -i \oint_{\partial D} f(\frac{z^2 + 1}{2z}, \frac{z^2 - 1}{2iz})\; \frac{1}{z}\; dz$$

kjer je $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$

2. naj bosta $p(t)$ in $q(t)$ polinoma, za katera je $st(q) \geq st(p) + 2$. za vsak dovolj velik R » 1 veljajo naslednje zveze

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)}\; dt = \oint_{\partial D_R} \frac{p(z)}{q(z)}\; dz$$

$$\int_{-\infty}^{\infty} \frac{p(t)cos(t)}{q(t)}\; dt = Re \oint_{\partial D_R} \frac{p(z)e^{iz}}{q(z)}\; dz$$

$$\int_{-\infty}^{\infty} \frac{p(t)sin(t)}{q(t)}\; dt = Im \oint_{\partial D_R} \frac{p(z)e^{iz}}{q(z)}\; dz$$

kjer je $D_R = \{z \in \mathbb{C} \mid |z| \leq R, Im(z) \geq 0\}$

Tabela odvodov

$f(x)$	$f'(x)$	$f(x)$	$f'(x)$	$f(x)$	$f'(x)$
x^n	nx^{n-1}	a^x	$a^x \ln(a)$	e^x	e^x
$\frac{1}{x^n}$	$-\frac{n}{x^{n-1}}$	$\ln x$	$\frac{1}{x}$	$\log_a x$	$\frac{1}{xlna}$
$\csc x$	$-\cot(x)\csc(x)$	$\cos x$	$-\sin x$	$\tan x$	$\frac{1}{\cos^2 x}$
$\sec x$	$\tan(x)\sec(x)$	$\sin x$	$\cos x$	$\cot x$	-$\frac{1}{\sin^2 x}$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$	$\tanh x$	$\frac{1}{\cosh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$	$\tanh^{-1} x$	$\frac{1}{1-x^2}$

Znane limite

$\lim_{x\to\infty} a^x = 0\ ,\ |a| < 1$ $\quad \lim_{x\to 0} x^x = 1$ $\quad \lim_{x\to\infty} \sqrt[x]{x} = 1$

$\lim_{x\to 0} \frac{a^x-1}{x} = \ln a\ ,\ a > 0$ $\quad \lim_{x\to\infty} \frac{\ln x}{x} = 0$ $\quad \lim_{x\to 0} x \ln x = 0$

$\lim_{x\to\infty} \left(1+\frac{k}{x}\right)^{mx} = e^{mk}$ $\quad \lim_{x\to 0} \left(1+kx\right)^{\frac{m}{x}} = e^{mk}$ $\quad \lim_{x\to 0} \frac{\sin x}{x} = 1$

nedoločene oblike
$\frac{0}{0} (L.H.), \frac{\infty}{\infty} (L.H.), 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$
prevladujoči členi
$n^n \gg n! \gg q^n\ (|q| > 1) \gg n^a\ (a > 0) \gg ln(n)^a\ (a > 0)$

Tabela integralov

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1}(n \neq -1)$	$\frac{1}{x}$	$\ln\|x\|$	e^x	e^x
$\sin x$	$-\cos x$	$\cos x$	$\sin x$	a^x	$\frac{a^x}{ln(a)}$
$\frac{1}{\cos^2 x}$	$\tan x$	$\frac{1}{\sin^2 x}$	$-\cot x$	$\cosh x$	$\sinh x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$	$\sinh^{-1} x$	$\frac{1}{1+x^2}$	$\arctan x$
$\sec x \tan x$	$\sec x$	$\csc x cotx$	$-\csc x$	$\tan x$	$\ln\|\sec x\|$
$\ln x$	$x \ln x - x$				

$\int \frac{dx}{\cos^2 ax} = \frac{\tan ax}{a} + C$ $\qquad \int \frac{dx}{\sin^2 ax} = -\frac{\cot ax}{a} + C$
$\int \frac{dx}{a+x} = \ln|a+x| + C$ $\qquad \int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan(\frac{x}{a}) + C$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin(\frac{x}{a}) + C$
$t = \tan(\frac{x}{2})$ $\quad \sin(x) = \frac{2t}{1+t^2}$ $\quad \cos(x) = \frac{1-t^2}{1+t^2}$ $\quad dx = \frac{2dt}{1+t^2}$
$t = \tan(x)$ $\quad \sin(x) = \frac{t}{\sqrt{1+t^2}}$ $\quad \cos(x) = \frac{1}{\sqrt{1+t^2}}$ $\quad dx = \frac{dt}{1+t^2}$

per partes $\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$
racionalne funkcije $\int \frac{p(x)}{q(x)}dx$
če je $st(q(x)) <= st(p(x))$: (1) polinoma delimo, (2) $q(x)$ razdelimo na linearne in kvadratne faktorje, (3) izraz pod integralom razcepimo na parcialne ulomke, (4) integriramo vsakega zase
kotne funkcije $\int \cos^m x \sin^n x dx$
če je eno od števil m,n **liho**, uporabimo tisti člen za t substitucijo
če sta obe **sodi**, jih nadomestimo z identiteto polovičnih kotov

Izrazi

$(a \pm b)^2 = a^2 \pm 2ab + b^2$
$(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
$a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
$a^n - b^n = (a-b)(a^{n-1} + \cdots + b^{n-1})$
$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$ $\qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$
$1 + a^{2n+1} = (1+a)(1-a+\cdots-a^{2n-1}+a^{2n})$

Potence, koreni, logaritmi

$a^n a^m = a^{n+m}$ $\quad a^n b^n = (ab)^n$ $\quad (a^n)^m = a^{nm}$ $\quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$
$\frac{a^n}{a^m} = a^{n-m}$ $\quad \frac{a^n}{b^n} = \left(\frac{a}{b}\right)^n$ $\quad a^{-n} = \frac{1}{a^n}$ $\quad ab^{-n} = \frac{a}{b^n}$
$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$ $\quad \frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ $\quad \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$ $\quad a^{\frac{m}{n}} = \sqrt[n]{a^m}$
$(-a)^{2n} = a^{2n}$ $\quad \sqrt[n]{a}\sqrt[n]{b} = \sqrt[n]{ab}$ $\quad (-a)^{2n+1} = -a^{2n+1}$
$\log_a x^n = nlog_a x$ $\quad log_b x = \frac{\log_a x}{log_a b}$ $\quad \log_a y = x \iff a^x = y$
$\log_a(xy) = \log_a(x) + \log_a(y)$ $\quad \log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$

Kompleksna števila

$\alpha = a + bi$ $\quad \bar{\alpha} = a - bi$ $\quad \alpha\beta = (ac - bd) + (ad + bc)i$
$a = \frac{\alpha+\bar{\alpha}}{2}$ $\quad b = \frac{\alpha-\bar{\alpha}}{2i}$ $\quad \frac{\beta}{\alpha} = \frac{\beta\bar{\alpha}}{|\alpha|^2}$ $\quad \alpha\bar{\alpha} = |\alpha|^2$
$|\alpha| = \sqrt{a^2+b^2}$ $\quad arg(\alpha) = atan2(a, b)$
$\alpha^n = |\alpha|^n e^{in\varphi}$ $\quad \alpha\beta = |\alpha|\,|\beta|\, e^{i(\varphi_{(\alpha)}+\varphi_{(\beta)})}$
$\alpha^n = |\alpha|^n \left(\cos(n\varphi) + i\sin(n\varphi)\right)$ $\quad \alpha^n = |\alpha|^n e^{i(n\varphi)}$
$\sqrt[n]{\alpha} = \sqrt[n]{|\alpha|}\left(\cos(\frac{\varphi}{n} + \frac{2\pi k}{n}) + i\sin(\frac{\varphi}{n} + \frac{2\pi k}{n})\right)$
$\sqrt[n]{\alpha} = \sqrt[n]{|\alpha|} e^{i(\frac{\varphi}{n} + \frac{2\pi k}{n})}$ $\quad {\scriptstyle k=0,1,2,\ldots,n-1}$

Kvadratna funkcija

$f(x) = ax^2 + bx + c$ $\quad f(x) = a(x-x_1)(x-x_2)$
$f(x) = a(x-p)^2 + q$ $\quad x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
teme T(p,q): $\quad p = -\frac{b}{2a}$ $\quad q = -\frac{b^2-4ac}{4a}$

Stožnice

parabola $\quad (y-q)^2 = \pm 2a(x-p)$ $\quad d(T, \prod) = \frac{|ax+by+cz-d|}{\sqrt{a^2+b^2+c^2}}$
krožnica $\quad (y-q)^2 = \pm 2a(x-p)$ $\quad \vec{p}\cdot\vec{q} = |\vec{p}|\,|\vec{q}|\cos\vartheta$
elipsa $\quad \frac{(x-p)^2}{a^2} + \frac{(y-q)^2}{b^2} = 1$
hiperbola $\quad \frac{(x-p)^2}{a^2} - \frac{(y-q)^2}{b^2} = \pm 1$

Obsegi, površine, volumni

tip	obseg	površina	tip	površina	volumen
krog	$2\pi r$	πr^2	krogla	$4\pi r^2$	$\frac{4\pi r^3}{3}$
enak.trik.	$3a$	$\frac{a^2\sqrt{3}}{4}$	tetraeder	$a^2\sqrt{3}$	$\frac{a^3\sqrt{2}}{12}$
trapez	$a+b+c+d$	$\frac{a+c}{2}h$	valj	$2\pi r(r+h)$	$\pi r^2 h$
deltoid	$2a+2b$	$ab\sin\alpha$	stožec	$2\pi r(r+s)$	$\frac{\pi r^2 h}{3}$

**h = height, s = slant*

Kotne funkcije

	$0°$	$30°$	$45°$	$60°$	$90°$					
	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	Q1	Q2	Q3	Q4	S/L
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	+	+	-	-	L
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	+	-	-	+	S
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	$-$	+	-	+	-	L
$\cot\alpha$	$-$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	+	-	+	-	L

$\sin\alpha = \frac{N}{H}$ $\quad \cos\alpha = \frac{P}{H}$ $\quad tan\alpha = \frac{N}{P}$ $\quad \cot\alpha = \frac{P}{N}$
$\sin^2\alpha = 1 - \cos^2\alpha$ $\quad \cos^2\alpha = 1 - \sin^2\alpha$
$\frac{1}{\sin^2\alpha} = 1 + \cot^2\alpha$ $\quad \frac{1}{\cos^2\alpha} = 1 + \tan^2\alpha$
$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1}{2}(1-\cos\alpha)}$ $\quad \tan\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha}$
$\cos\frac{\alpha}{2} = \pm\sqrt{\frac{1}{2}(1+\cos\alpha)}$ $\quad \cot\frac{\alpha}{2} = \frac{\sin\alpha}{1-\cos\alpha}$
$2\sin^2\frac{\alpha}{2} = 1 - \cos\alpha$ $\quad 2\cos^2\frac{\alpha}{2} = 1 + \cos\alpha$
$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}$ $\quad \cos^2\alpha = \frac{1+\cos 2\alpha}{2}$
$\sin 2\alpha = 2\sin\alpha\cos\alpha$ $\quad \cos 2\alpha = 2\cos\alpha - 2\sin\alpha$
$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$ $\quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$
$\sin(\alpha\pm\beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$
$\cos(\alpha\mp\beta) = \cos\alpha\cos\beta \pm \sin\alpha\sin\beta$
$\tan(\alpha\pm\beta) = \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta}$
$\sin\alpha \pm \sin\beta = 2\sin\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$
$\cos\alpha \pm \cos\beta = 2\cos\frac{\alpha\pm\beta}{2}\cos\frac{\alpha\mp\beta}{2}$
$\cos\alpha\cos\beta = \frac{1}{2}(\cos(\alpha+\beta) + \cos(\alpha-\beta))$
$\sin\alpha\cos\beta = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$
$\sin\alpha\sin\beta = -\frac{1}{2}(\cos(\alpha+\beta) - \cos(\alpha-\beta))$
$\sin\alpha - \cos\beta = -2\sin(\frac{\alpha+\beta}{2})\sin(\frac{\alpha-\beta}{2})$

Hiperbolične funkcije

$\sinh x$	$\frac{e^x-e^{-x}}{2} = \frac{e^{2x}-1}{2e^x} = \frac{1-e^{-2x}}{2e^{-x}}$	
$\cosh x$	$\frac{e^x+e^{-x}}{2} = \frac{e^{2x}+1}{2e^x} = \frac{1+e^{-2x}}{2e^{-x}}$	
$\sinh^{-1} x$	$\ln(x+\sqrt{x^2+1})$	
$\cosh^{-1} x$	$\ln(x+\sqrt{x^2-1})$	$x \geq 1$
$\tanh^{-1} x$	$\frac{1}{2}\ln(\frac{1+x}{1-x})$	$\|x\| < 1$
$\coth^{-1} x$	$\frac{1}{2}\ln(\frac{x+1}{x-1})$	$\|x\| > 1$

$\cosh x + \sinh x = e^x$ $\quad \cosh^2 x - \sinh^2 x = 1$
$\cosh x - \sinh x = e^{-x}$
**ident. kotnih funkcij, vendar se pri sinh(x) * sinh(y) obrne predznak*

Krožne funkcije

$\sin^{-1} x$	$D_f = [-1,1]$	$Z_f = [-\frac{\pi}{2},\frac{\pi}{2}]$
$\cos^{-1} x$	$D_f = [-1,1]$	$Z_f = [0,\pi]$
$\tan^{-1} x$	$D_f = (-\infty,\infty)$	$Z_f = (-\frac{\pi}{2},\frac{\pi}{2})$
$\cot^{-1} x$	$D_f = (-\infty,\infty)$	$Z_f = (0,\pi)$
$\sec^{-1} x$	$D_f = (-\infty,-1] \cup [1,\infty)$	$Z_f = [0,\frac{\pi}{2}) \cup (\frac{\pi}{2},\pi]$
$\csc^{-1} x$	$D_f = (-\infty,-1] \cup [1,\infty)$	$Z_f = [-\frac{\pi}{2},0) \cup (0,\frac{\pi}{2}]$

Uporabne vrste

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad x \in \mathbb{R}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \qquad x \in \mathbb{R}$$

$$\sinh(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad x \in \mathbb{R}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad x \in \mathbb{R}$$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 \cdots \qquad x \in (-1,1)$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 \cdots \qquad x \in (-1,1)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \qquad x \in (-1,1]$$

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} \cdots \qquad x \in [-1,1)$$

Opomba: $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\ldots(\alpha-(n-1))}{n!}$